Cycles, Chords, and Planarity in Graphs

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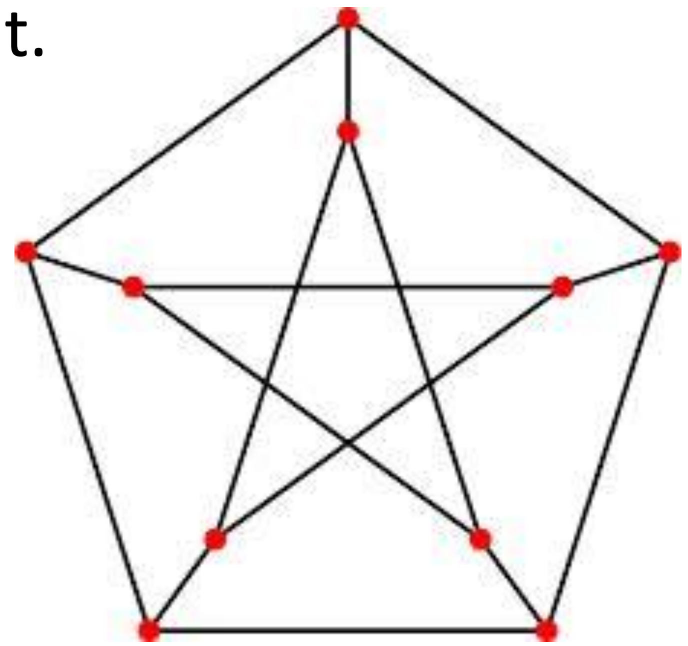
A 501 project based on Etienne Birmelé's article "Every Longest Circuit of a 3-Connected, $K_{3,3}$-Minor Free Graph Has a Chord"

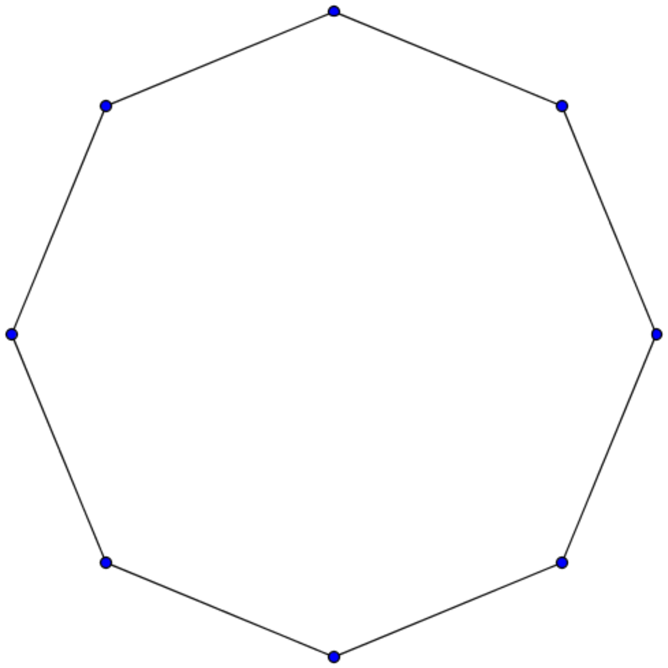
A *graph* G is a pair of sets, one of vertices V, and one of edges E, along with a relation that associates edges with pairs of vertices. These vertices are its *endpoints* and two vertices are *adjacent (neighbors)* if they are endpoints of the same edge. A graph is *simple* if it has no loops or multiple edges. The *degree* of a vertex is the number of edges to which it is an endpoint.



The Petersen Graph with 10 vertices and 15 edges.

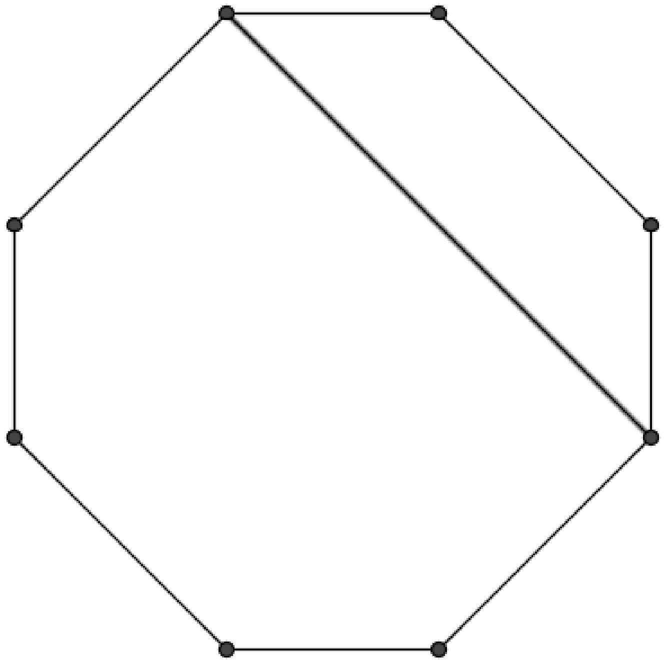
Path

Cycle



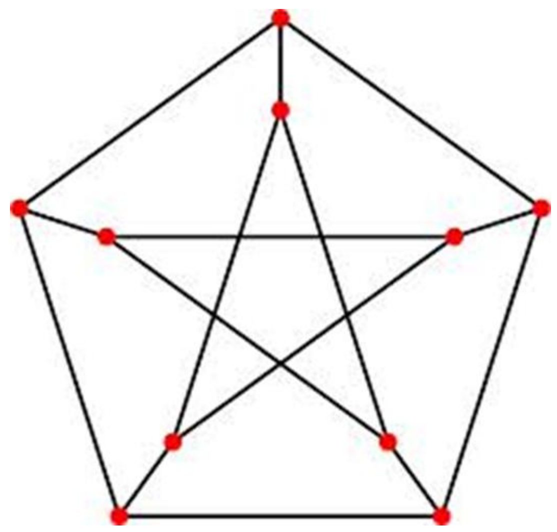
Chords

A cycle has a *chord* if there are a pair of vertices that are adjacent, but not along the cycle.



Connected and k-Connected

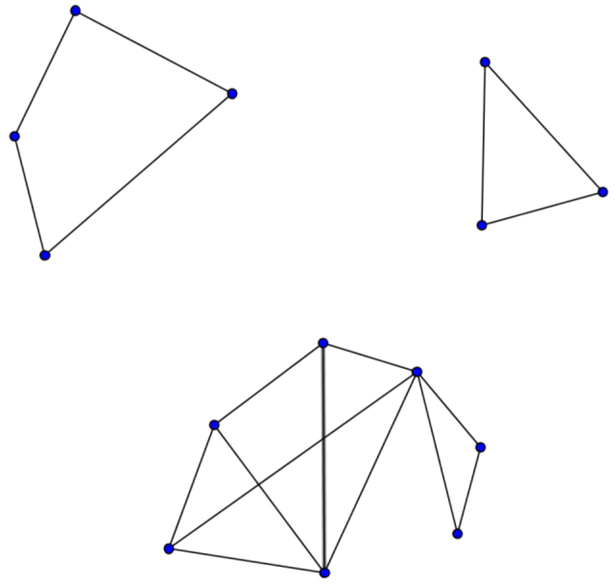
A graph G is *connected* if for any pair of vertices u, v, there is a path in G that has u and v as endpoints. G is *k-connected* if the removal of any set of k vertices from G results in a graph that is neither disconnected or a single vertex. (Specifically, a connected graph is 0-connected).



The Petersen graph is 2-connected, but not 3-connected

Components

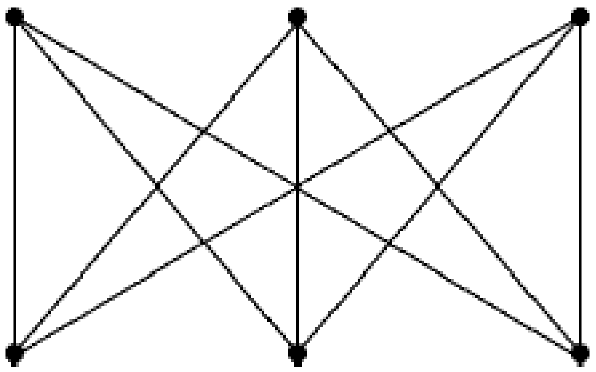
The *components* of a graph are its maximal connected subgraphs.



This graph has 3 components

Bipartite Graphs and $K_{n,m}$

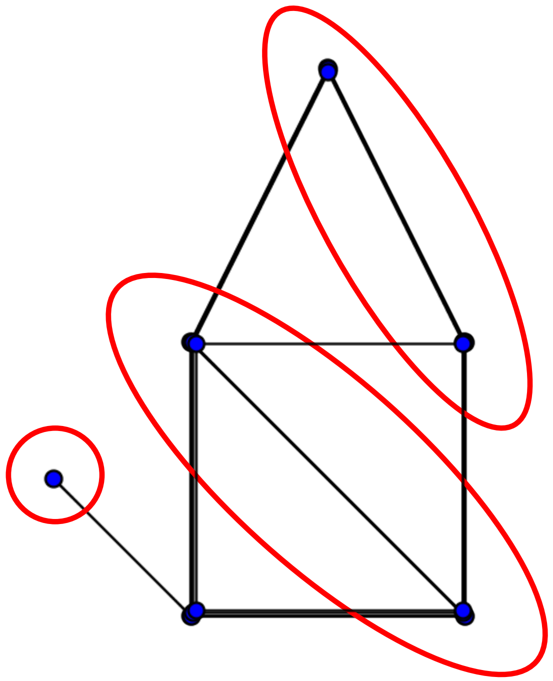
A *bipartite* graph is a graph where the vertices can be partitioned into two disjoint subsets such that each subset contains no pairwise adjacent vertices. The graph $K_{n,m}$ is the complete (all edges) bipartite graph where one partition has n vertices and the other m.



The graph $K_{3,3}$

Graph Minors

H is a *minor* of a graph G if H is obtainable from G by a sequence of vertex and edge deletions and edge contractions.



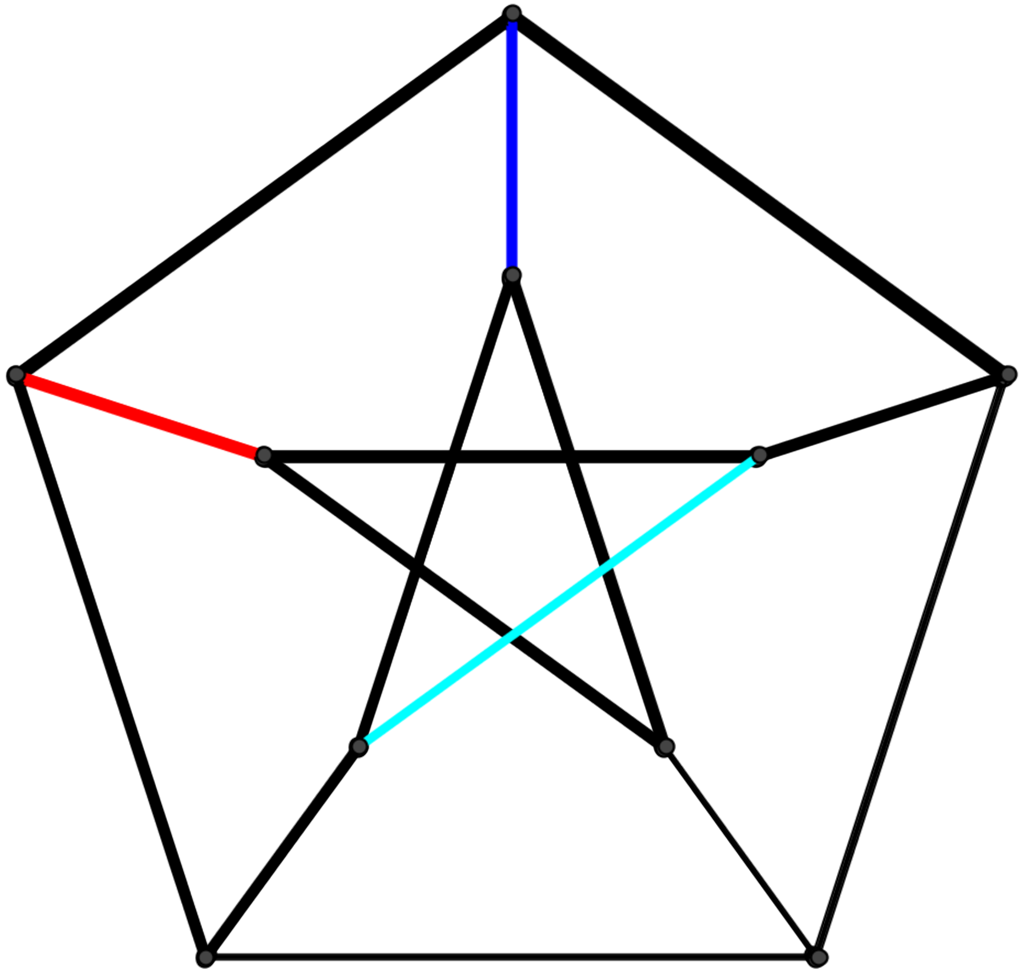
A cycle with 4 vertices is a minor of the starting graph

Theorem

In 1982, noted Graph Theorist Carsten Thomassen conjectured that every longest cycle of a 3-connected graph has a chord.

Thus we have this theorem, a significant milestone toward finding the truth of this conjecture, by Etienne Birmelé:

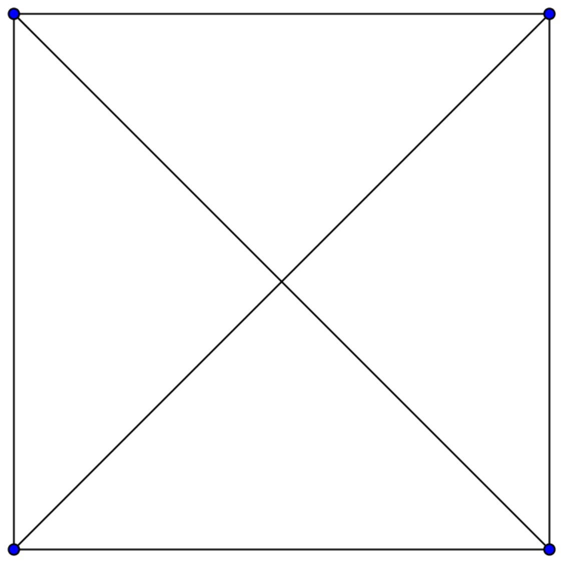
Every longest [cycle] of a 3-connected, $K_{3,3}$-minor free graph has a chord.



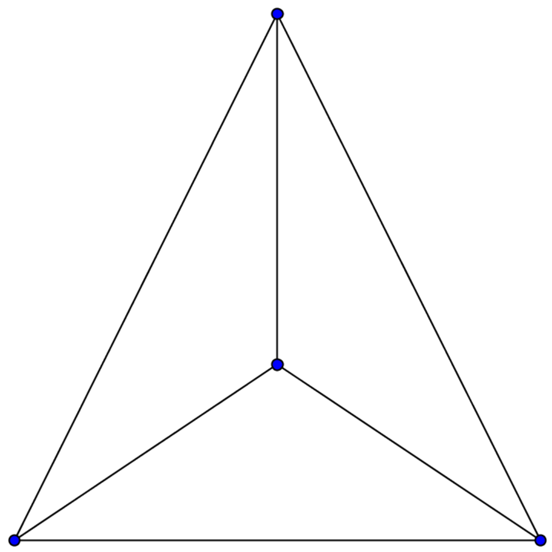
A longest length cycle of the Petersen graph has multiple chords

Planar Graphs

A graph is called *planar* if it can be represented on the plane without crossed edges.

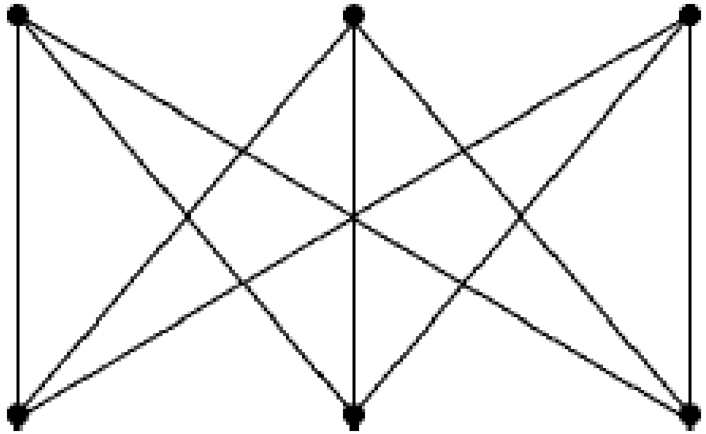


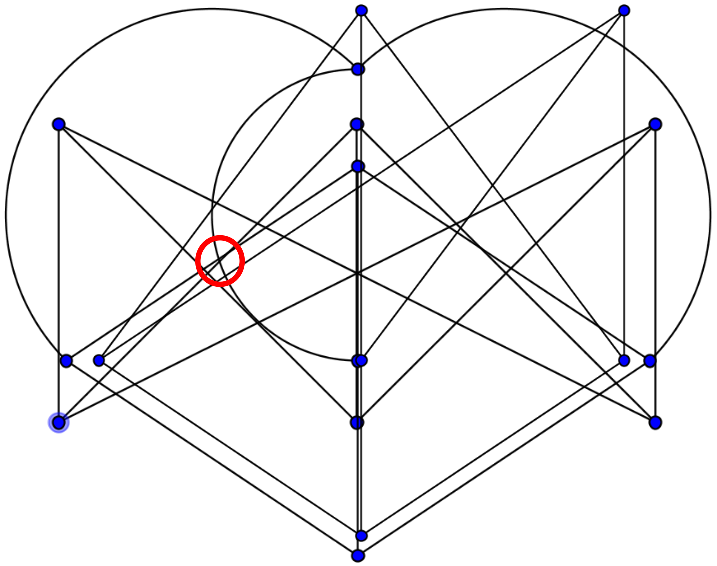
K_4 may seem to be nonplanar here…



But because we **can** draw it without crossed edges, it is planar!

However, $K_{3,3}$ is **not** planar. No matter how it’s depicted, it will always have at least one crossed edge.





Wagner’s Theorem

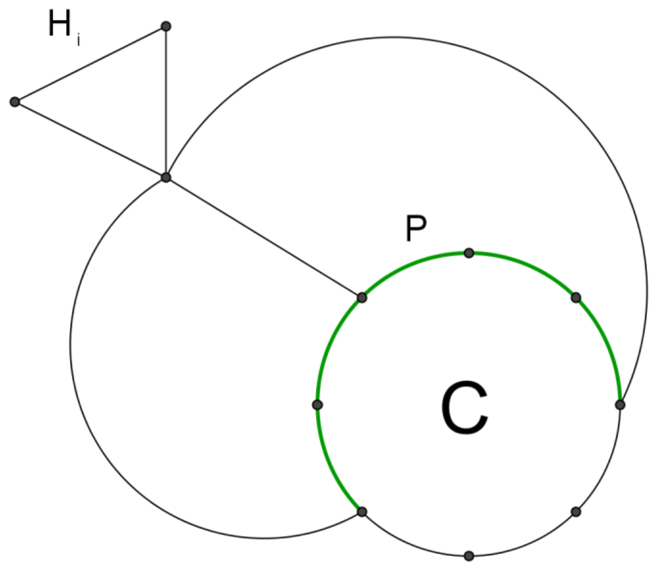
In 1937, Klaus Wagner published this theorem named for him:

A graph is planar if and only if it contains no K_5 and no $K_{3,3}$ minor.

Therefore, our theorem comes with the corollary that every longest cycle of a 3-connected, planar graph has a chord.

Component Support

We begin the proof by selecting a longest length cycle C. We then look at G \ C, and let $H_1,\ldots,H_r$ be the connected components of G \ C. We denote by N(i) the vertices on C that are adjacent to some vertex in H_i. Let P be an arc (connected subgraph) of C. We say P is a *support* of H_i if N(i) is a subset of V(P).



Lemma 4 (i) and (ii)

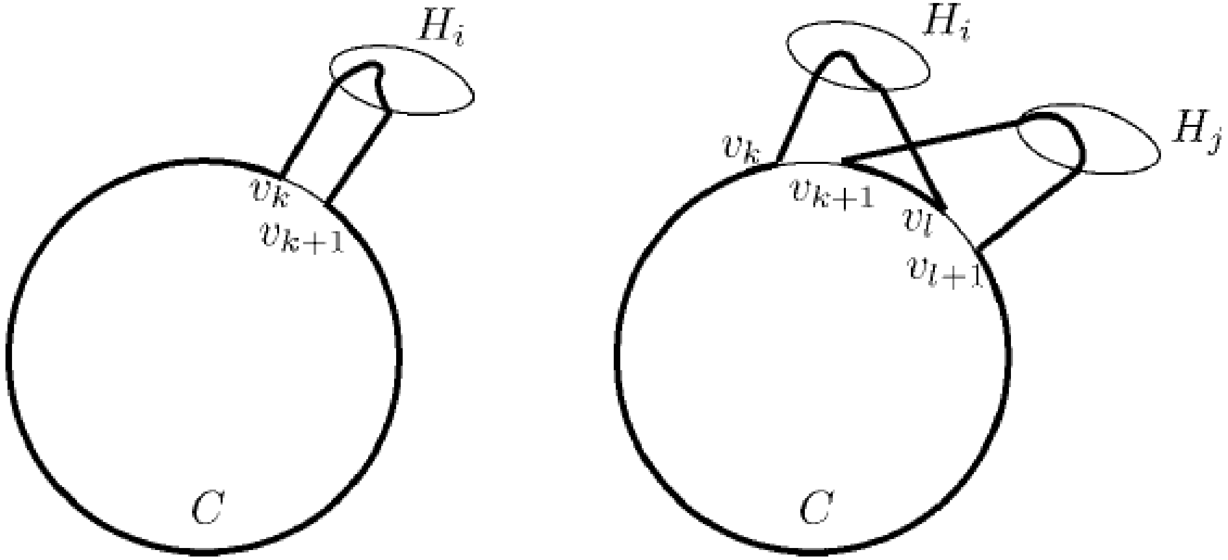
(i) Every N(i) has at least three vertices.
(ii) Every vertex of the cycle is in at least one N(i).

Each of these points are thanks to the fact that G is 3-connected. If some N(i) have one or two vertices, then H_i could be disconnected by the removal of fewer than three vertices. Also, every vertex on C has two neighbors in C, and so must have one neighbor in some H_i.

Lemma 4 (iii) and (iv)

(iii) Two consecutive vertices of C cannot belong to the same N(i).

(iv) There is no pair of consecutive vertices of C such that the first of each pair belongs to an N(i) and the second of each belongs to an N(j).



Not possible because C is a longest length cycle.

How Do We Solve This?

We do so by looking at supports that are minimal with respect to inclusion. We start by selecting one minimal support P of some H_i (and we will call P without its endpoints P'). Then, we find a support Q for some H_j such that:

i) N(j) and P' share at least one vertex
ii) P ∪ Q is minimal with respect to inclusion
iii) Subject to the first two conditions, Q is minimal with respect to inclusion

Vertices of Interest

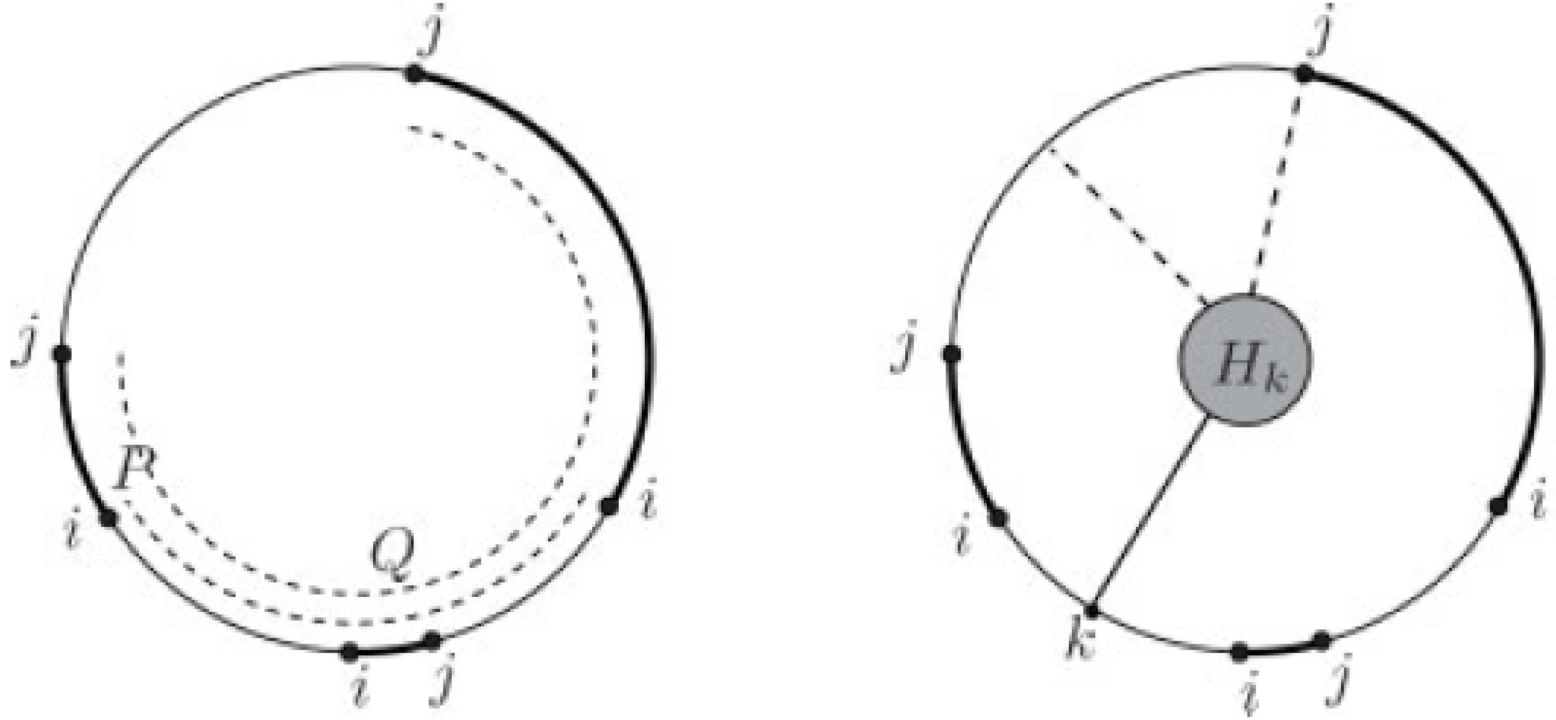
We then look at six (not necessarily totally distinct) vertices we will call *vertices of interest*:

- Three distinct vertices of N(i). Two of these three are the two endpoints of P.

- Three distinct vertices of N(j). Two vertices are the endpoints of Q, and one must be on P’ (the vertex on P’ may or may not be an endpoint of Q).

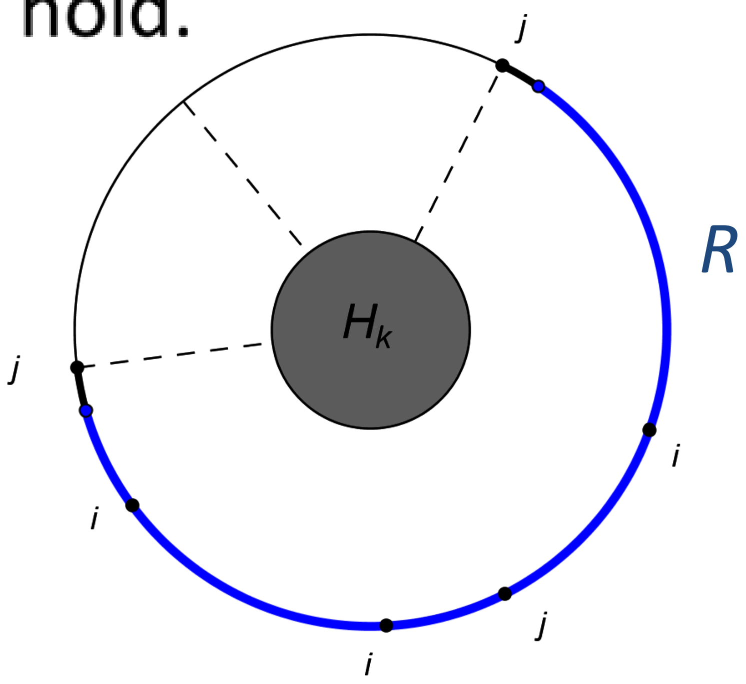
Now For a Case Breakdown

We now divide the proof into a number of cases. The first case we look at is when P is a subpath of Q.



How Do We Know One of The Edges Exists?

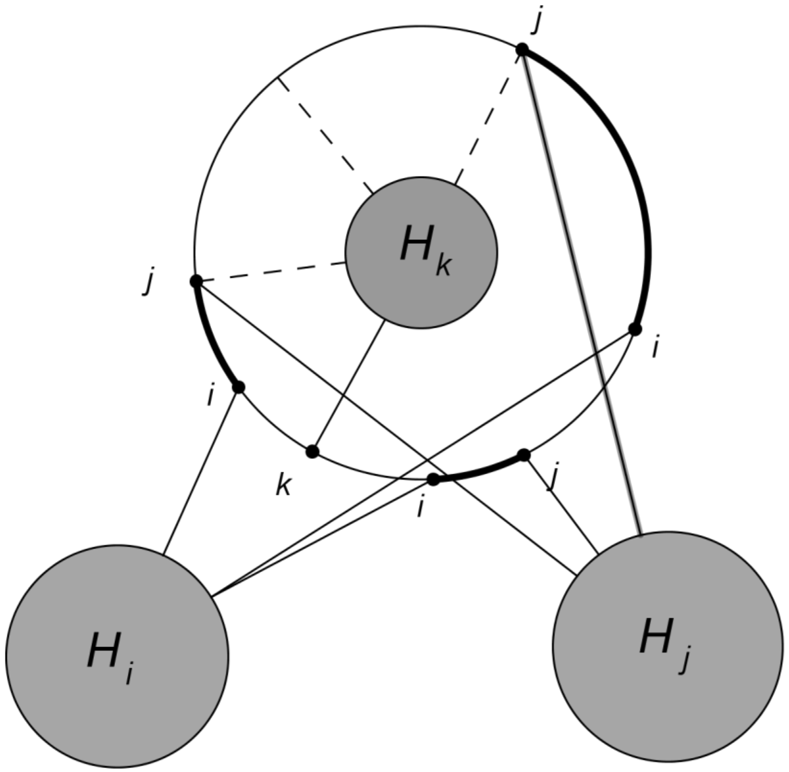
If there is no vertex from N(k) outside P ∪ Q, and both endpoints of Q are not in N(k), then we find that one of our minimality assumptions doesn't hold.



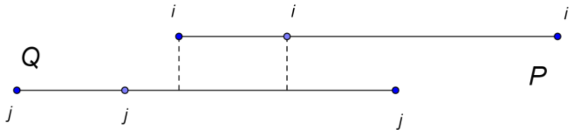
Either P ∪ R is a proper subset of P ∪ Q **or** R is a proper subset of Q

Finding a $K_{3,3}$ Minor

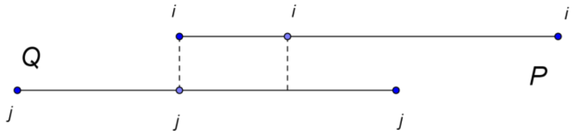
We then contract the bold arcs and H_i and H_j to single points, as well as H_k along with any vertices in N(k) not on a bold arc to a single vertex.



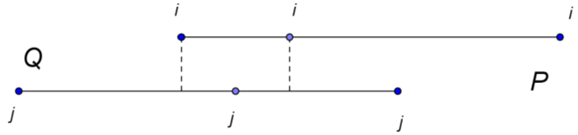
Other Cases (P is not a subpath of Q)



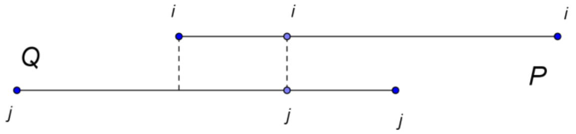
Case 2



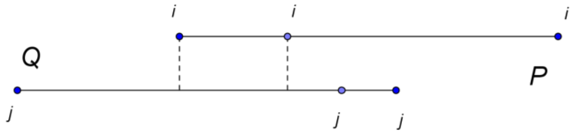
Case 3



Case 4

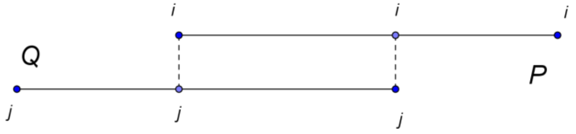


Case 5

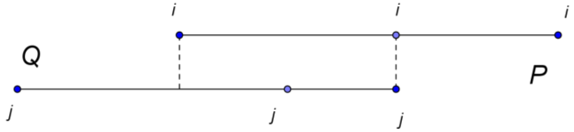


Case 6

Case 7



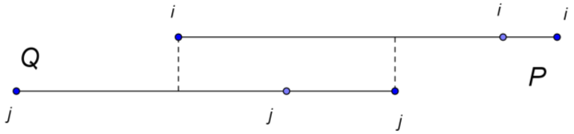
Case 8



Case 9

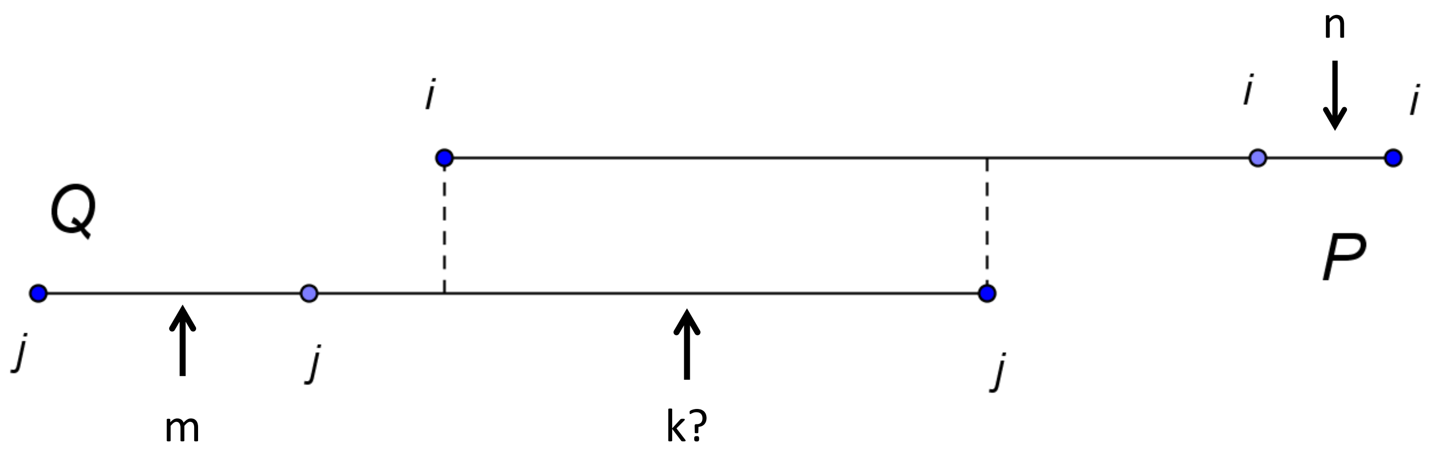
Case 10

Case 11



Case 12

Trouble Case (Case 10)



We can't guarantee a k in the middle…

And while we can find an m and n on the outside, we don't know if $H_m = H_n$…

Conclusion

Using the 3-connectedness of a graph, along with the assumption that it contains no $K_{3,3}$-minor, Birmelé finds that assuming the lack of a chord leads to a contradiction, the presence of a $K_{3,3}$-minor. This is a powerful step toward solving Thomassen's conjecture. With further research into the absent case, this would give us a large collection of graphs for which the conjecture holds, or a possible counterexample.

References

1. Etienne Birmelé, Every Longest Circuit of a 3-Connected, $K_{3,3}$-Minor Free Graph Has a Chord.
2. Douglas B. West, Introduction to Graph Theory, Second Edition, Prentice-Hall Inc., Upper Saddle River, NJ, 1996.

3. Wagner’s Theorem, Wikipedia, http://en.wikipedia.org/wiki/Wagner%27s_theorem